

AN INVERSE PROBLEM FOR A HYPERBOLIC SYSTEM ON A VECTOR BUNDLE AND ENERGY MEASUREMENTS

KATSIARYNA KRUPCHYK AND MATTI LASSAS

ABSTRACT. Consider a second order hyperbolic linear system with time independent coefficients whose solution and source term for each time moment are sections of a real smooth vector bundle over a closed connected smooth manifold. Assume that the energy of the system is conserved. Moreover, assume that the manifold together with its topological and differential structures and a smooth measure on it are known and we are given sets of sources which are known to be localized in space and time and dense in some function spaces. By measuring how much energy is needed to produce different combinations (i.e. sums) of these sources, we reconstruct a vector bundle up to an isomorphism, the Riemannian structure on it and the system. We also show that these sets of sources are generic in the sense that they can be almost surely constructed by taking some sequences of realizations of suitable independent identically distributed Gaussian random variables.

1. INTRODUCTION

In this paper we consider a smooth real vector bundle V with a Riemannian structure over a compact closed connected smooth manifold M and a second order hyperbolic initial value problem

$$\begin{cases} (\partial_t^2 + A(x, \partial_x))u(x, t) = F(x, t), & x \in M, \quad t > -\tau, \\ u(x, t)|_{t=-\tau} = 0, \quad \partial_t u(x, t)|_{t=-\tau} = 0, \end{cases}$$

with some large $\tau > 0$ and $\text{supp}(F) \subset M \times (-\tau, +\infty)$. Here for each time moment $t \in \mathbb{R}$ the solution $u(\cdot, t) = u^F(\cdot, t) : M \rightarrow V$ and the source term $F(\cdot, t) : M \rightarrow V$ are sections of the vector bundle V .

Let an elliptic operator $A = A(x, \partial_x)$ be such that the energy of the wave u^F produced by the source F , defined by formula

$$E_A(F, t) = \frac{1}{2} \int_M (\langle \partial_t u^F(x, t), \partial_t u^F(x, t) \rangle_x + \langle Au^F(x, t), u^F(x, t) \rangle_x) dV(x),$$

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is conserved, that is, if $F = 0$ for all $t \geq t_0$ then $E_A(F, t) = E_A(F, t_0)$. Here $\langle \cdot, \cdot \rangle_x$ is the inner product in the fiber $\pi^{-1}(x)$ of the vector bundle V assigned by the Riemannian structure and $dV(x)$ is a smooth measure on M .

Assume that the manifold M together with its topological and differential structures and a smooth measure $dV(x)$ on it are known. The inverse problem studied in this paper is to reconstruct the vector bundle V , the Riemannian structure on it and the hyperbolic system by measuring energy of waves u^{F_j} , as functions $t \mapsto E_A(F_j, t)$, for certain family of sources F_j , $j = 1, 2, 3, \dots$.

Since the vector bundle is not known, we encounter immediately the problem how to specify the sources F_j which we use in the measurements. To approach this problem, we assume that in some method we can produce the families of sources of the following type:

- (i) $C^\infty(M \times \bar{I})$ -dense sequence of sources F_j , $j = 1, 2, \dots$, on $M \times \bar{I}$ where $I \subset \mathbb{R}$ is an open interval;
- (ii) $C(\bar{U} \times \bar{I})$ -dense sequence of sources F_j , $j = 1, 2, \dots$, on $\bar{U} \times \bar{I}$ where U is an open set on M .

These families of sources are generic in the sense that they can be almost surely generated by taking some sequences of realizations of suitable independent identically distributed Gaussian random variables, see Section 3. Roughly speaking, we assume that we can just produce a family of physical sources supported in the set $\bar{U} \times \bar{I}$ but we do not know what the functions describing these sources are. We call such produced sources the basic experiments. We also assume that we can combine disjointly supported basic experiments and delay them in time and measure how much energy is needed to perform the combined experiments. Using this data, we want to determine the physical model, that is, the bundle and the coefficients of the hyperbolic system.

Similar problems of recovering the operator $A(x, \partial_x)$ and the Riemannian manifold from energy measurements have been encountered in the inverse boundary value problems. The prototype inverse problem is the inverse conductivity problem, called also the Calderón's inverse problem (see [6]). Consider a conductivity equation

$$\begin{aligned} \nabla \cdot \sigma(x) \nabla u(x) &= 0 \quad \text{in } D, \\ u|_{\partial D} &= f, \end{aligned} \tag{1.1}$$

where $D \subset \mathbb{R}^n$ is smooth domain, $\sigma(x)$ is a positive-definite matrix corresponding to the conductivity at the point x , u is the electric potential having boundary value f . The inverse conductivity problem means the determination of σ from measurements done at the boundary ∂D . One possibility to define the boundary measurements is the following (there are many equivalent formulations): assume

that for all boundary values f we measure

$$Q_\sigma(f) = \int_D (\sigma(x) \nabla u(x)) \cdot \nabla u(x) dx.$$

This is equivalent to measuring the power needed to keep the voltage u at the boundary ∂D to be equal to $f(x)$. Physically, the boundary value $u|_{\partial D} = f$ is produced by electrodes attached on the boundary of the body, and $Q_\sigma(f)$ corresponds to the power of the heat produced by the caused currents in the domain D . Because of conservation of energy, the power produced by the heat is equal to the power needed to keep the electrodes at the specified voltages. When σ is equal to a scalar function $\gamma(x)$ times the identity matrix, the conductivity is called isotropic. In this case the measurements are shown to determine the conductivity – the problem was solved for smooth conductivities in [24, 20] and in the form posed by Calderón in [6] for L^∞ conductivity in two dimensions in [2], see also [5, 13, 16].

When the conductivity σ in (1.1) is matrix valued (physically, the conductivity is anisotropic), it has been shown that the conductivity σ can not be determined uniquely and that the best one can hope is to determine σ up to a change of coordinates. This has been shown in dimension 2 in [3, 22, 23]. In dimensions $n \geq 3$ determining the conductivity is equivalent to determining the isometry type of a Riemannian metric corresponding to the conductivity, and this is a longstanding open problem. For related works, see [10, 11, 14, 18, 19].

As another example of inverse problems on a Riemannian manifold which motivates our study, let us consider the wave equation,

$$\begin{cases} (\partial_t^2 - \Delta_g)u(x, t) = 0 & (x, t) \in M \times \mathbb{R}_+, \\ u|_{t=0} = 0, \quad \partial_t u|_{t=0} = 0, \\ u|_{\partial M \times \mathbb{R}_+} = f \end{cases} \quad (1.2)$$

on a compact Riemannian manifold (M, g) with boundary ∂M . Assume that we know the boundary ∂M and can measure for all $f \in C_0^\infty(\partial M \times \mathbb{R}_+)$ the final energy $Q_g(f)$ of the waves u produced by f , that is,

$$Q_g(f) = \lim_{t \rightarrow \infty} \frac{1}{2} \int_M (|\nabla_g u(x, t)|_g^2 + |\partial_t u(x, t)|^2) dV_g(x).$$

By energy conservation, this can be considered as the total energy needed to force the boundary value $u|_{\partial M \times \mathbb{R}_+}$ to be equal f . When $Q_g(f)$ is known for all $f \in C_0^\infty(\partial M \times \mathbb{R}_+)$, then the isometry type of the Riemannian manifold (M, g) can be recovered [15], see also [1, 14].

In modern physics, phenomena are often modeled by PDE systems on sections of vector bundles. The vector bundles are encountered from the early development of the relativistic quantum mechanics to modern quantum field theory. Hence the study of inverse problems on general vector bundles brings the mathematical

study of inverse problems closer to modern physics. A result in this direction is [17] where the inverse problem for the Dirac equation on a general vector bundle over a compact Riemannian manifold with nonempty boundary is studied.

Let us consider also other examples, analogous to the problem studied in this paper, where reconstruction of the bundle structure is interesting. Consider the collision of elementary particles. When such collisions are produced, the precise parameters determining the collision are not usually known, but one just observes that a collision has happened and its consequences can be also observed. Such non-exactly known experiments are analogous to the measurements considered in this paper. By observing behavior of electrons and other particles the physicists deduced the existence of the spin structures, which implies that vector bundles with spin structures are important in the large scale models in a curved universe. A priori, before the physical observations, it is not clear what the bundle structure is. There are physical models for yet unobserved particles, such as Dirac's magnetic monopole, which are based on the use of non-trivial bundles. Thus, if in the future one would observe a new particle behaving like a magnetic monopole, and the results of the physical experiments for that would coincide with those predicted by Dirac's magnetic monopole model, one should ask if there are other models giving the same physical predictions. The reconstruction problem of the bundle structure studied in this paper can be considered as a highly idealized model problem for such a physical problem.

Reconstruction of unknown "bundle structure" appears also in much more mundane problems than in the high energy physics. When making observations on a chemical system where reactions happen in several compartments (e.g. in different cells and blood in a metabolic model [7, 8]), we can often observe only part of the chemical concentrations and only at some compartments. The diffusion-reaction equation can be considered as an equation on a vector bundle, where the structure of the bundle corresponds to the knowledge, how many substances are present in the reactions, and the differential equation corresponds to the knowledge what kind of reactions happen in each compartment. Similarly, in mathematical finance, in modeling prices on a stock market, one can look prices as observations and ask how many hidden variables are needed to be added to obtain a linear stochastic differential equation that fit well to observations. There, the observed prices and the hidden variables can be considered as a vector bundle.

There are many other practical examples where the measurements can not exactly be modeled. For instance in geophysical measurements implemented using explosives or air guns in water, the model for measurement process is not very exact. Thus the methodology of this paper, the combining of a set of measurements which are not well known, can be useful for improving reconstruction techniques with such inaccurately known measurements.

The rest of the paper is organized as follows. In Section 2 we present some facts related to vector bundles, rigorously formulate our basic assumptions on sources used in energy measurements and state our main result. In Section 3 we explain that our assumptions on sources are quite natural by constructing such sources as sequences of realizations of suitable independent identically distributed Gaussian random variables. Section 4 is devoted to reconstruction of a bundle, the Riemannian structure on it and the system from measuring how much energy is needed to produce different combinations of the sources.

2. DEFINITIONS AND MAIN RESULTS

2.1. Real vector bundles. Let M be a closed connected smooth m -dimensional manifold. Let V be a smooth real vector bundle over M and denote by $\pi : V \rightarrow M$ the projection onto the base manifold. Each fiber $\pi^{-1}(x)$ has the structure of a real vector space isomorphic to \mathbb{R}^d and a Riemannian structure that assigns a smooth inner product $\langle \cdot, \cdot \rangle_x$.

Given two vector bundles $\pi_1 : V_1 \rightarrow M$ and $\pi_2 : V_2 \rightarrow M$, a map $\Phi : V_1 \rightarrow V_2$ is a vector bundle homomorphism if $\pi_1 = \pi_2 \Phi$ and Φ restricts on each fiber to a linear map $\pi_1^{-1}(x) \rightarrow \pi_2^{-1}(x)$. A bundle homomorphism $\Phi : V_1 \rightarrow V_2$ with an inverse which is also a bundle homomorphism is called a vector bundle isomorphism, and then V_1 and V_2 are said to be isomorphic vector bundles. If Φ preserves the Riemannian structure, then Φ is called an isometry.

Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas of a vector bundle V , i.e. $\{U_\alpha\}$ is an open cover of M and $\phi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^d$ are local trivializations. For any two vector bundle charts (U_α, ϕ_α) and (U_β, ϕ_β) such that $U_\alpha \cap U_\beta \neq \emptyset$, the composition $\phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^d \rightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^d$ must be of the form $\phi_\alpha \circ \phi_\beta^{-1}(x, v) = (x, t_{\alpha\beta}(x)v)$ for some smooth map $t_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(d, \mathbb{R})$ which is called a transition map. For a given vector bundle atlas the family of transition maps always satisfy the following properties:

$$\begin{aligned} t_{\alpha\alpha}(x) &= \text{Id for all } x \in U_\alpha \text{ and all } \alpha; \\ t_{\alpha\beta}(x) \circ t_{\beta\alpha}(x) &= \text{Id for all } x \in U_\alpha \cap U_\beta; \\ t_{\alpha\gamma}(x) \circ t_{\gamma\beta}(x) \circ t_{\beta\alpha}(x) &= \text{Id for all } x \in U_\alpha \cap U_\beta \cap U_\gamma. \end{aligned} \tag{2.1}$$

A family of maps $\{t_{\alpha\beta}\}$, $t_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(d, \mathbb{R})$ which satisfies (2.1) for some cover of M is called a $\text{GL}(d, \mathbb{R})$ -cocycle.

The following theorem gives the minimal information required to reconstruct a vector bundle up to an isomorphism.

Theorem 2.1. [21, Section 9.2.2] *Given a manifold M , a cover $\{U_\alpha\}$ of M and a $\text{GL}(d, \mathbb{R})$ -cocycle $\{t_{\alpha\beta}\}$ for the cover, there exists a vector bundle with an atlas*

$\{(U_\alpha, \phi_\alpha)\}$ satisfying $\phi_\alpha \circ \phi_\beta^{-1}(x, v) = (x, t_{\alpha\beta}(x)v)$ on nonempty overlaps $U_\alpha \cap U_\beta$. A vector bundle constructed in such a way is unique up to an isomorphism.

2.2. Function spaces. Denote by $C^\infty(M, V)$ the space of smooth sections of the bundle V . The inner product in $L^2(M, V)$ is defined by

$$\langle \phi, \psi \rangle = \int_M \langle \phi(x), \psi(x) \rangle_x dV(x)$$

where $dV(x)$ is a smooth measure on M . For the vector bundle V over M , we consider also a related vector bundle \tilde{V} over $M \times \mathbb{R}$. When V has the transition functions $t_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(d, \mathbb{R})$, the bundle \tilde{V} has the transition functions $\tilde{t}_{\alpha\beta} : (U_\alpha \times \mathbb{R}) \cap (U_\beta \times \mathbb{R}) \rightarrow GL(d, \mathbb{R})$, $\tilde{t}_{\alpha\beta}(x, t) = t_{\alpha\beta}(x)$, $(x, t) \in M \times \mathbb{R}$. Let $L^2(M \times \mathbb{R}, \tilde{V})$ and $H^s(M \times \mathbb{R}, \tilde{V})$ be the L^2 and Sobolev spaces of sections of the vector bundle \tilde{V} , respectively. We often write $L^2(M \times \mathbb{R}, \tilde{V}) = L^2(M \times \mathbb{R})$, etc.

Now let U be an open set on M and I be an open interval in \mathbb{R} . Then we set

$$\begin{aligned} H^s(U \times I) &= \{F|_{U \times I} : F \in H^s(M \times \mathbb{R})\}, \quad s \in \mathbb{R}, \\ \tilde{H}^s(U \times I) &= \{F \in H^s(M \times \mathbb{R}) : \text{supp } (F) \subset \overline{U \times I}\}, \quad s \in \mathbb{R}. \end{aligned}$$

For the spaces $L^2(U \times I)$, $H^s(U \times I)$ and $C^\infty(M \times \bar{I})$, sometimes we identify functions with their zero continuations in $M \times \mathbb{R}$. We recall the duality

$$(H^s(U \times I))' = \tilde{H}^{-s}(U \times I).$$

2.3. Formulation of the problem and main result. Let

$$A : C^\infty(M, V) \rightarrow C^\infty(M, V), \quad A = \sum_{|\alpha| \leq 2} a_\alpha(x) \partial_x^\alpha,$$

be an elliptic self-adjoint positive second-order partial differential operator on sections of real vector bundle over a closed smooth manifold. In local trivializations, $a_\alpha(x)$ are matrices. Consider the hyperbolic initial value problem

$$\begin{cases} (\partial_t^2 + A(x, \partial_x))u(x, t) = F(x, t), & x \in M, \quad t > -\tau, \\ u(x, t)|_{t=-\tau} = 0, \quad \partial_t u(x, t)|_{t=-\tau} = 0, \end{cases} \quad (2.2)$$

with some large $\tau > 0$ and $\text{supp } (F) \subset M \times (-\tau, +\infty)$. Let us denote by $u^F(x, t)$ the solution to (2.2).

The energy of the wave u^F is defined by

$$E_A(F, t) = \frac{1}{2} \int_M (\langle \partial_t u^F(x, t), \partial_t u^F(x, t) \rangle_x + \langle A u^F(x, t), u^F(x, t) \rangle_x) dV(x)$$

where dV is a smooth measure on M . Since the operator A is self-adjoint, the energy is conserved, that is, if $\text{supp } (F) \subset M \times [t_0^-, t_0^+]$ and $t', t'' > t_0^+$ then $E_A(F, t') = E_A(F, t'')$.

Let $\mathcal{B}_M = \{U_p, p = 1, 2, \dots\}$ be a basis of topology of M and assume that $U_1 = M$. Let $\mathcal{B}_{\mathbb{R}} = \{I_q, q = 1, 2, \dots\}$ be a basis of topology on \mathbb{R} consisting of open intervals. By *basic experiments* we mean producing the following sets of sources:

(BE) for any $U_p \in \mathcal{B}_M$ and $I_q \in \mathcal{B}_{\mathbb{R}}$, we produce a set of sources $\mathcal{F}_{U_p, I_q} = \{F_k^{p,q}, k = 1, 2, \dots\}$ supported on $\overline{U_p} \times \overline{I_q}$ which is dense in $C(\overline{U_p} \times \overline{I_q})$, if $p \neq 1$, or in $C^\infty(M \times \overline{I_q})$, if $p = 1$.

In Section 3 we show that the basic experiments can be almost surely generated by taking some sequences of realizations of suitable independent identically distributed Gaussian random variables.

Denote by $(\tau_T F)(x, t) = F(x, t + T)$ the time shift for a source F . Let $J \in \mathbb{N}$ and $F_j \in \mathcal{F}_{U_{p(j)}, I_{q(j)}}$, $U_{p(j)} \in \mathcal{B}_M$, $I_{q(j)} = (t_{q(j)}^-, t_{q(j)}^+) \in \mathcal{B}_{\mathbb{R}}$, $j = 1, \dots, J$. Assume that for any $0 \leq t \in \mathbb{Q}$ and $0 \leq T_j \in \mathbb{Q}$, we can measure energy

$$E_A\left(\sum_{j=1}^J \tau_{T_j} F_j, t\right)$$

provided that the time intervals $[t_{q(j)}^- - T_j, t_{q(j)}^+ - T_j]$ are disjoint. These measurements are called *admissible measurements*. This means that we can combine separate basic experiments, delay the sources in time, and measure how much energy is used in the combined experiments.

In the paper using admissible energy measurements for the sources generated by the basic experiments, we give a method to recover a bundle V over the known manifold M (up to an isomorphism), the Riemannian structure $\langle \cdot, \cdot \rangle_x$ on it and an operator A . As our reconstruction procedure consists of infinite number of steps we formulate our main result as an uniqueness result.

To simplify the notations, for countable bases \mathcal{B}_M and $\mathcal{B}_{\mathbb{R}}$, we renumber the sequences of sources generated by the basic experiments using one index, i.e. $F_j = F_{k(j)}^{p(j), q(j)}$, $j \in \mathbb{N}$. Letting $K \subset \mathbb{N}$ be a finite set, we denote

$$E_A(K, (T_j)_{j=1}^\infty, t) = E_A\left(\sum_{j \in K} \tau_{T_j} F_j, t\right).$$

Thus, we have the following uniqueness result.

Theorem 2.2. *Assume that a closed smooth manifold M together with its topological and differential structures and a smooth measure on M are given. Let V and \tilde{V} be real smooth vector bundles with Riemannian structures over M and $A(x, \partial_x)$ and $\tilde{A}(x, \partial_x)$ be elliptic self-adjoint positive second-order partial differential operators on sections of V and \tilde{V} , respectively. Let $(F_j)_{j=1}^\infty = (F_{k(j)}^{p(j), q(j)})_{j=1}^\infty$*

and $(\tilde{F}_j)_{j=1}^\infty = (\tilde{F}_{k(j)}^{p(j),q(j)})_{j=1}^\infty$ be sequences of sources generated by the basic experiments for V and \tilde{V} , correspondingly. If for any $K \subset \mathbb{N}$, $(T_j)_{j=1}^\infty$, $0 \leq T_j \in \mathbb{Q}$ and $0 \leq t \in \mathbb{Q}$,

$$E_A(K, (T_j)_{j=1}^\infty, t) = E_{\tilde{A}}(K, (T_j)_{j=1}^\infty, t),$$

then there is an isometry $\Phi : V \rightarrow \tilde{V}$ such that $\tilde{A} = \Phi A \Phi^{-1}$.

Roughly speaking, Theorem 2.2 states that if we do measurements using combinations (i.e., sums) of some unknown basic sources $(F_j)_{j=1}^\infty$, and find the energies needed to produce all combinations of these sources, then the physical model (i.e., the Riemannian bundle structure and the operator) is uniquely determined up to an isometry.

3. GENERATION OF BASIC EXPERIMENTS. RANDOM SOURCES

The purpose of this section is to prove that for any $I \in \mathcal{B}_{\mathbb{R}}$ and $U \in \mathcal{B}_M$, sequences of sources, which are dense in $C^\infty(M \times \overline{I})$ and $C(\overline{U} \times \overline{I})$, can be almost surely generated by taking some sequences of realizations of suitable independent identically distributed Gaussian random variables. This shows that the sequences of sources we consider are generic.

Let $(\Omega, \Sigma, \mathbb{P})$ be a complete probability space. Recall the notion of a Gaussian random variable taking values in a real separable Hilbert space Y with inner product $\langle \cdot, \cdot \rangle_Y$. Let us identify the dual of Y with Y using Riesz representation theorem. Let \mathcal{N}_Y be the Borel σ -algebra of (Y, τ^w) with τ^w being the weak topology of Y . Note that the separability of Y verifies that \mathcal{N}_Y coincides with the Borel σ -algebra of (Y, τ^n) , where τ^n is the norm topology of Y . An Y -valued random variable F is a measurable map $F : (\Omega, \Sigma) \rightarrow (Y, \mathcal{N}_Y)$. In this paper we consider only random variables with values in separable Hilbert spaces. We say that F is *Gaussian*, if for any $\psi \in Y$ the real-valued functions $\omega \mapsto \langle \psi, F(\omega) \rangle_Y$ are Gaussian random variables. Then Y -valued random variable has the *expectation* $\mathbb{E} F \in Y$ and the *covariance operator* $C_F = C_{F,Y} : Y \rightarrow Y$ (in the space Y) which are defined by

$$\langle \mathbb{E} F, \psi \rangle_Y = \mathbb{E} \langle F, \psi \rangle_Y, \quad \text{for } \psi \in Y,$$

and

$$\mathbb{E} (\langle F - \mathbb{E} F, \phi \rangle_Y \langle F - \mathbb{E} F, \psi \rangle_Y) = \langle C_F \phi, \psi \rangle_Y, \quad \text{for } \phi, \psi \in Y. \quad (3.1)$$

Note that if Y_1 and Y_2 are two separable Hilbert spaces so that $Y_1 \subset Y_2$, any Y_1 -valued gaussian random variable F can be also considered as Y_2 -valued random variable but the covariance operator of F in the space Y_1 , C_{F,Y_1} , do not generally coincide with the covariance operator of F in the space Y_2 , C_{F,Y_2} , but

$$C_{F,Y_2} = J C_{F,Y_1} J^*$$

where $J : Y_1 \rightarrow Y_2$ is the identical embedding and $J^* : Y_2 \rightarrow Y_1$ is its adjoint. The covariance operator $C_F : Y \rightarrow Y$ is always a selfadjoint non-negative trace class operator in Y [4, Theorem 2.3.1].

Let F be a Gaussian random variable in Y and let $C_F : Y \rightarrow Y$ have the eigenvalues $\lambda_j \geq 0$ and let φ_j , $j = 1, 2, \dots$ be the orthonormal eigenfunctions which form a basis in Y . Then F can be written in the form

$$F(\omega) = \sum_{j=1}^{\infty} a_j G_j(\omega) \varphi_j, \quad \omega \in \Omega,$$

where $G_j = \langle F, \varphi_j \rangle_Y$, $G_j : \Omega \rightarrow \mathbb{R}$ are independent normalized Gaussian random variables and $a_j = \sqrt{\lambda_j}$.

Lemma 3.1. *Let F_j , $j = 1, 2, \dots$, be independent identically distributed Gaussian Y -valued random variables with $\mathbb{E}F_j = 0$ and covariance operators $C_{F_j} : Y \rightarrow Y$ being injective. Then the set $\{F_j(\omega), j = 1, 2, \dots\}$ is almost surely dense in Y with weak topology.*

Proof. To see the claim one has to show that for any open set $W \neq \emptyset$ in weak topology,

$$\mathbb{P}(\cup_{j=1}^{\infty} (F_j \in W)) = 1.$$

Let $T_{z,r,\varepsilon} = \{y \in Y : |\langle z, y \rangle_Y - r| < \varepsilon\}$ for $z \in Y$, $r \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}_+$. Then the set

$$B_{basis} = \{\cap_{i=1}^N T_{z_i, r_i, \varepsilon_i} : z_i \in Y, r_i \in \mathbb{R}, \varepsilon_i \in \mathbb{R}_+, N \in \mathbb{N}\}$$

is a basis of (Y, τ^w) .

Let us consider an arbitrary set $A \in B_{basis} \setminus \{\emptyset\}$, i.e.

$$A = \bigcap_{i=1}^N T_{z_i, r_i, \varepsilon_i}, \quad z_i \in Y, \quad r_i \in \mathbb{R}, \quad \varepsilon_i \in \mathbb{R}_+, \quad N \in \mathbb{N} \quad (3.2)$$

Let us next consider a single random variable F_j . Our first goal is to prove that

$$\mathbb{P}(F_j \in A) > 0. \quad (3.3)$$

Let $P : Y \rightarrow Y$ be an orthogonal projector onto a finite dimensional space $Z = \text{span}(z_1, \dots, z_N)$ and endow Z with the same inner product as Y .

Since the covariance operators $C_{F_j} : Y \rightarrow Y$ is injective, $C_{F_j} : Y \rightarrow Y$ has the following positivity property:

$$\text{if } \psi \in Y \text{ satisfies } \langle C_{F_j} \psi, \psi \rangle_Y = 0, \text{ then } \psi = 0. \quad (3.4)$$

Then PF_j is Z -valued random variable having zero expectation and covariance operator

$$C_{PF_j} : Z \rightarrow Z, \quad C_{PF_j} = PC_{F_j}P.$$

Now if $\psi \in Z$ is such that $\langle C_{PF_j}\psi, \psi \rangle_Z = 0$, then

$$0 = \langle C_{PF_j}\psi, \psi \rangle_Z = \langle PC_{F_j}P\psi, \psi \rangle_Z = \langle C_{F_j}P\psi, P\psi \rangle_Y.$$

As $P\psi = \psi$, we have $\langle C_{F_j}\psi, \psi \rangle_Y = 0$ and thus $\psi = 0$. This implies also $C_{PF_j} : Z \rightarrow Z$ satisfies the positivity property (3.4).

Now as A is open and non-empty, $PA \subset Z$ is open and non-empty. Identifying Z with \mathbb{R}^N , (3.4) yields that C_{PF_j} is an invertible matrix and thus, we have

$$\mathbb{P}(PF_j \in PA) = c \int_{PA} \exp(-\frac{1}{2}x \cdot C_{PF_j}^{-1}x) dx > 0.$$

By definition of P and A we get that $F_j \in A$ if and only if $PF_j \in PA$. Hence,

$$\mathbb{P}(F_j \in A) = \mathbb{P}(PF_j \in PA) > 0$$

that shows the claim (3.3).

Now let $\{y_i\}_{i=1}^\infty$ be a dense set in (Y, τ^n) with norm topology and consider the enumerable set

$$B^{enum} = \{\cap_{i=1}^N T_{y_i, r_i, \varepsilon_i} : y_i \in \{y_i\}_{i=1}^\infty, r_i \in \mathbb{Q}, \varepsilon_i \in \mathbb{Q}_+, N \in \mathbb{N}\}.$$

Denote by $B_R = \{y \in Y : \|y\|_Y < R\}$ the ball of the radius R in Y and introduce enumerable sets

$$\begin{aligned} \mathcal{V}_1 &= \{V \cap B_R : R \in \mathbb{Q}_+, V \in B^{enum}\}, \\ \mathcal{V}_2 &= \{\cup_{i=1}^n A_i : A_i \in \mathcal{V}_1, n \in \mathbb{N}\}, \\ \mathcal{V}_3 &= \{V \in \mathcal{V}_2 : \mathbb{P}(F_j \in V) > 0\}. \end{aligned}$$

Now let \tilde{A} be a set with the following property

$$\tilde{A} \in \mathcal{V}_1 \quad \text{and} \quad \tilde{A} \subset A. \tag{3.5}$$

Let us number all the sets with the property (3.5) by A_1, A_2, \dots . Then

$$A = \bigcup_{i=1}^\infty A_i.$$

We set

$$V_n = \bigcup_{i=1}^n A_i,$$

and thus, $V_n \subset V_{n+1}$, $V_n \in \mathcal{V}_2$ and $V_n \subset A$. By the monotone convergence theorem,

$$\lim_{n \rightarrow \infty} \mathbb{P}(F_j \in V_n) = \mathbb{P}(F_j \in A) > 0.$$

Hence, for a non-empty set A of type (3.2), there exists a set $V \in \mathcal{V}_3$ such that $V \subset A$.

By Kolmogorov's 0-1 law, we have for any $V \in \mathcal{V}_3$

$$\mathbb{P}(\cup_{j=1}^{\infty} (F_j \in V)) = 1, \quad \text{i.e.} \quad \mathbb{P}([\cup_{j=1}^{\infty} (F_j \in V)]^c) = 0.$$

Since the set \mathcal{V}_3 is enumerable, we get

$$0 = \sum_{V \in \mathcal{V}_3} \mathbb{P}([\cup_{j=1}^{\infty} (F_j \in V)]^c) \geq \mathbb{P}(\cup_{V \in \mathcal{V}_3} [\cup_{j=1}^{\infty} (F_j \in V)]^c)$$

Thus, we have

$$1 = \mathbb{P}(\cap_{V \in \mathcal{V}_3} \cup_{j=1}^{\infty} (F_j \in V)) = \mathbb{P}(\{\forall V \in \mathcal{V}_3 \exists j \in \mathbb{N} : F_j \in V\}).$$

As any non-empty weakly open set W in Y contains a non-empty set A of the type (3.2), and any such set A contains a set $V \in \mathcal{V}_3$, this yields

$$\mathbb{P}(\{\forall W \in \tau^w, W \neq \emptyset, \exists j \in \mathbb{N} : F_j \in W\}) = 1.$$

Thus the set $\{F_j : j = 1, 2, \dots\}$ is almost surely weakly dense in Y .

□

As compact operators map weakly convergent sequences to norm convergent sequences, we obtain the following corollary.

Corollary 3.2. *Let F_j , $j \in \mathbb{N}$, be independent identically distributed Gaussian Y -valued random variables with $\mathbb{E}F_j = 0$ and covariance operators $C_{F_j} : Y \rightarrow Y$ being injective. Let \tilde{Y} be a Hilbert space so that $Y \hookrightarrow \tilde{Y}$ is compact and $Y \subset \tilde{Y}$ is dense. Then the set $\{F_j(\omega), j = 1, 2, \dots\}$ is almost surely dense in \tilde{Y} in norm topology.*

Next we give an example how one can construct a set of the Gaussian random variables which is weakly dense in $H^s(M \times \mathbb{R})$ for any $s \geq 0$. First recall that the Gaussian random variable is determined uniquely by its expectation and covariance operator. An operator $C_{F, H^s} : H^s(M \times \mathbb{R}) \rightarrow H^s(M \times \mathbb{R})$ is a covariance operator of some random variable F if and only if C_{F, H^s} is self-adjoint non-negative trace class operator in $H^s(M \times \mathbb{R})$, see [4, Theorem 2.3.1].

Example 3.3. *Let us construct the Gaussian random variable F which can be considered as a random variable in the Sobolev space $H^s(M \times \mathbb{R})$ with an arbitrary $s \geq 0$ such that $\mathbb{E}F = 0$ and it has an injective covariance operator. First assume that $s = 0$ and consider a Schödinger operator*

$$L = (-\Delta_g - \partial_t^2 + t^2) : L^2(M \times \mathbb{R}) \rightarrow L^2(M \times \mathbb{R}).$$

It is known [9] that the spectrum of L is discrete. We set $C_{F, H^0} = e^{-L}$.

Now assume that $s > 0$ and let G be a Gaussian random variable in $H^s(M \times \mathbb{R})$ with the covariance operator $C_{G, H^s} : H^s(M \times \mathbb{R}) \rightarrow H^s(M \times \mathbb{R})$ and $\mathbb{E}G = 0$.

Let $J : H^s(M \times \mathbb{R}) \rightarrow L^2(M \times \mathbb{R})$, $J\phi = \phi$, be the embedding. Then the random variable $JG : (\Omega, \Sigma) \rightarrow L^2(M \times \mathbb{R})$ has the covariance operator

$$C_{JG, L^2} = JC_{G, H^s}J^*. \quad (3.6)$$

One can see that $J^* = K^{-s} : L^2(M \times \mathbb{R}) \rightarrow H^s(M \times \mathbb{R})$ where $K = (-\Delta - \partial_t^2 + 1)$. Now setting $C_{JG, L^2} = e^{-L}$, (3.6) implies that $C_{G, H^s} = e^{-L}K^{-s} : H^s(M \times \mathbb{R}) \rightarrow H^s(M \times \mathbb{R})$ which is clearly injective.

Let F_j , $j = 1, 2, \dots$ be independent identically distributed Gaussian $H^s(M \times \mathbb{R})$ -valued random variables, $s \geq 0$, with $\mathbb{E}F_j = 0$ and injective covariance operators $C_{F_j} : H^s(M \times \mathbb{R}) \rightarrow H^s(M \times \mathbb{R})$. Then Lemma 3.1 the set $\{F_j(\omega), j = 1, 2, \dots\}$ is almost surely weakly dense in $H^s(M \times \mathbb{R})$ for all $s \geq 0$. Let $I \in \mathcal{B}_{\mathbb{R}}$. Then $\{F_j(\omega)|_{M \times I}, j = 1, 2, \dots\}$ is almost surely weakly dense in $H^s(M \times I)$ for all $s \geq 0$. Consider $s_0, s > s_0 > 0$. Then by Corollary 3.2, $\{F_j(\omega)|_{M \times I}, j = 1, 2, \dots\}$ is almost surely dense in $H^{s_0}(M \times I)$ with norm topology for any $s_0 > 0$, and thus in $C^\infty(M \times \bar{I})$ with a Fréchet topology. Recall that the space $C^\infty(M \times \bar{I})$ is a Fréchet space with the topology determined by the family of seminorms $p_k(F) = \|F\|_{C^k(M \times \bar{I})}$, $k = 0, 1, 2, \dots$.

Let $U \in \mathcal{B}_M$. Then

$$\{F_j(\omega)|_{U \times I}, j = 1, 2, \dots\} \quad (3.7)$$

is almost surely dense in $C(\bar{U} \times \bar{I})$. Also, the set (3.7) is almost surely dense in $L^2(U \times I)$ with norm topology.

4. PROOF OF THEOREM 2.2.

4.1. Measurements. For convenience of the presentation, we use the following notations. Let $I_1 = (t_1^-, t_1^+)$ and $I_2 = (t_2^-, t_2^+)$ be intervals in \mathbb{R} and $\tau_0 \in \mathbb{R}$. We write

$$\begin{aligned} I_1 < I_2 & \quad \text{if} \quad t_1^+ < t_2^-, \\ I_1 < \tau_0 & \quad \text{if} \quad t_1^+ < \tau_0. \end{aligned}$$

In this section we discuss what kind of information can be computed using admissible measurements. For this end, we need the following lemma.

Lemma 4.1. (i) Let $F_i \in C(\bar{U}_i \times \bar{I}_i)$, $U_i \in \mathcal{B}_M$, $I_i \in \mathcal{B}_{\mathbb{R}}$, $i = 1, 2$ (recall that we identify F_i with their zero continuations). Then for any time τ_0 ,

$$E_A(F_1 \pm F_2, \tau_0) \asymp \|u^{F_1}(\tau_0) \pm u^{F_2}(\tau_0)\|_{H^1(M, V)}^2 + \|\partial_t u^{F_1}(\tau_0) \pm \partial_t u^{F_2}(\tau_0)\|_{L^2(M, V)}^2. \quad (4.1)$$

(ii) Let $F_i \in C^\infty(M \times \bar{I}_i)$, $I_i \in \mathcal{B}_\mathbb{R}$, $i = 1, 2$. Then for any τ_0 , $I_i < \tau_0$, $i = 1, 2$, and $s = 1, 2, \dots$,

$$E_A(\partial_t^s(F_1 \pm F_2), \tau_0) \asymp \|u^{F_1}(\tau_0) \pm u^{F_2}(\tau_0)\|_{H^{s+1}(M,V)}^2 + \|\partial_t u^{F_1}(\tau_0) \pm \partial_t u^{F_2}(\tau_0)\|_{H^s(M,V)}^2, \quad (4.2)$$

where $f_1 \asymp f_2$ if there are constants $c_1, c_2 > 0$ such that $c_1 f_1 \leq f_2 \leq c_2 f_1$.

Proof. The definition of the energy immediately implies (4.1). Let us prove (4.2). As $I_i < \tau_0$, we have $\partial_t^2 u^{F_i}(\tau_0) = -A u^{F_i}(\tau_0)$ and moreover,

$$(\partial_t)^{2k} u^{F_j}(\tau_0) = (-1)^k A^k u^{F_j}(\tau_0), \quad k = 1, 2, \dots \quad (4.3)$$

As A is an elliptic second order operator and $\lambda = 0$ is not its eigenvalue, we get

$$\begin{aligned} \|A^k u\|_{L^2(M,V)}^2 &\asymp \|u\|_{H^{2k}(M,V)}^2, \quad k = 1, 2, \dots, \\ \langle A(A^k u), A^k u \rangle &\asymp \|A^k u\|_{H^1(M,V)}^2 \asymp \|u\|_{H^{2k+1}(M,V)}^2, \quad k = 0, 1, 2, \dots \end{aligned} \quad (4.4)$$

Now for $s = 2k + 1$, $k = 0, 1, \dots$, (4.3) and (4.4) imply

$$\begin{aligned} E_A((\partial_t)^s(F_1 - F_2), \tau_0) &= \frac{1}{2} (\langle A(A^k(\partial_t u^{F_1}(\tau_0) - \partial_t u^{F_2}(\tau_0))), A^k(\partial_t u^{F_1}(\tau_0) - \partial_t u^{F_2}(\tau_0)) \rangle \\ &\quad + \|A^{k+1}(u^{F_1}(\tau_0) - u^{F_2}(\tau_0))\|_{L^2(M,V)}^2) \\ &\asymp \|\partial_t u^{F_1}(\tau_0) - \partial_t u^{F_2}(\tau_0)\|_{H^{2k+1}(M,V)}^2 + \|u^{F_1}(\tau_0) - u^{F_2}(\tau_0)\|_{H^{2k+2}(M,V)}^2. \end{aligned}$$

For $s = 2k$, $k = 0, 1, \dots$, (4.3) and (4.4) yield

$$\begin{aligned} E_A((\partial_t)^s(F_1 - F_2), \tau_0) &= \frac{1}{2} (\langle A(A^k(u^{F_1}(\tau_0) - u^{F_2}(\tau_0))), A^k(u^{F_1}(\tau_0) - u^{F_2}(\tau_0)) \rangle \\ &\quad + \|A^k \partial_t(u^{F_1}(\tau_0) - u^{F_2}(\tau_0))\|_{L^2(M,V)}^2) \\ &\asymp \|u^{F_1}(\tau_0) - u^{F_2}(\tau_0)\|_{H^{2k+1}(M,V)}^2 + \|\partial_t u^{F_1}(\tau_0) - \partial_t u^{F_2}(\tau_0)\|_{H^{2k}(M,V)}^2. \end{aligned}$$

□

In a sequel to prove that convergence of sequences of sources $F_j \rightarrow F$ when $j \rightarrow \infty$ implies convergence of waves $u^{F_j}(T_0) \rightarrow u^F(T_0)$ and $\partial_t u^{F_j}(T_0) \rightarrow \partial_t u^F(T_0)$ when $j \rightarrow \infty$ in appropriate spaces we need the following estimates.

Lemma 4.2. (i) Let either $F \in C^\infty(M \times \bar{I}_0)$ and $0 \leq s \in \mathbb{R}$ or $F \in C(\bar{U} \times \bar{I}_0)$ and $s = 0$. Then for any T_0 ,

$$\begin{aligned} \|u^F(T_0)\|_{H^{s+1}(M,V)} &\leq C_{I_0, T_0} \|F\|_{L^2(\bar{I}_0, H^{s+1}(M,V))}, \\ \|\partial_t u^F(T_0)\|_{H^s(M,V)} &\leq C_{I_0, T_0} \|F\|_{L^2(\bar{I}_0, H^s(M,V))}. \end{aligned} \quad (4.5)$$

- (ii) Let $F \in L^2(\bar{I}_0, H^1(M, V))$, $I_0 = (t_0^-, t_0^+)$. Then the function $t \rightarrow E_A(F, t)$ is continuous on any finite time interval $I_1 = (t_0^-, t_1^+)$.

Proof. The proof of (i) is quite well-known for hyperbolic systems but we include it for convenience of the reader. As A is an elliptic self-adjoint operator, there is an orthonormal basis of eigensections $\{\varphi_l\}$ of A in $L^2(M, V)$. Denote by $\lambda_l > 0$ an eigenvalue of A which corresponds to φ_l . Thus, the wave u^F can be represented by the Fourier series

$$u^F(x, t) = \sum_{l=1}^{\infty} u_l^F(t) \varphi_l(x), \quad u_l^F(t) = \langle u^F(\cdot, t), \varphi_l \rangle \quad (4.6)$$

with coefficients satisfying

$$u_l^F(t) = \int_{-\tau}^t \frac{\sin(\sqrt{\lambda_l}(t-s))}{\sqrt{\lambda_l}} F_l(s) ds, \quad F_l(s) = \langle F(\cdot, s), \varphi_l \rangle. \quad (4.7)$$

Thus,

$$|u_l^F(T_0)|^2 \leq C_{I_0, T_0} \int_{t_0^-}^{t_0^+} |F_l(s)|^2 ds.$$

Hence, we have

$$\begin{aligned} \|u^F(T_0)\|_{H^{s+1}(M, V)}^2 &= \sum_{l=1}^{\infty} (1 + \lambda_l^{s+1}) |u_l^F(T_0)|^2 \\ &\leq C_{I_0, T_0} \int_{t_0^-}^{t_0^+} \left(\sum_{l=1}^{\infty} (1 + \lambda_l^{s+1}) |F_l(s)|^2 \right) ds \\ &= C_{I_0, T_0} \int_{t_0^-}^{t_0^+} \|F(s)\|_{H^{s+1}(M, V)}^2 ds = C_{I_0, T_0} \|F\|_{L^2(\bar{I}_0, H^{s+1}(M, V))}^2. \end{aligned}$$

The second estimate in (4.5) can be proven in the same way. Now uniform convergence of series (4.6) and (4.7) implies (ii). \square

Remark 4.3. Lemma 4.2 implies that measuring energy $E_A(\sum_{j \in K} \tau_{T_j} F_j, t)$ for $T_j \in \mathbb{Q}$ and $t \in \mathbb{Q}$, we can find the energy $E_A(\sum_{j \in K} \tau_{T_j} F_j, t)$ for any $T_j \in \mathbb{R}$ and $t \in \mathbb{R}$.

The lemma below shows that for any source F defined for a time interval I_0 , there exist sequences of sources generated in the basic experiments and defined for a time interval I_1 satisfying $I_0 < I_1$ such that the sequences of the corresponding waves approximate the waves u^F and u^{-F} at time $\tau_0 > I_1$.

Lemma 4.4. Let either $F \in C^\infty(M \times \bar{I}_0)$ and $0 \leq s \in \mathbb{R}$ or $F \in C(\bar{U} \times \bar{I}_0)$ and $s = 0$. Then for F and any τ_0 , $I_0 < \tau_0$, there are sequences of sources

$F_j, G_j \in \mathcal{F}_{M,I_1}$, $I_1 \in \mathcal{B}_{\mathbb{R}}$, $j = 1, 2, \dots$, $I_0 < I_1 < \tau_0$, such that

$$\lim_{j \rightarrow \infty} (u^{F_j}(\tau_0), \partial_t u^{F_j}(\tau_0)) = (u^{-F}(\tau_0), \partial_t u^{-F}(\tau_0)) \quad \text{in } H^{s+1}(M, V) \times H^s(M, V). \quad (4.8)$$

$$\lim_{j \rightarrow \infty} (u^{G_j}(\tau_0), \partial_t u^{G_j}(\tau_0)) = (u^F(\tau_0), \partial_t u^F(\tau_0)) \quad \text{in } H^{s+1}(M, V) \times H^s(M, V). \quad (4.9)$$

Proof. Consider first the case when $F \in C^\infty(M \times \bar{I}_0)$. Let $\psi \in C^\infty(\mathbb{R})$ be such a function that $\psi(t) = 0$, when $t \leq t_1^-$, and $\psi(t) = 1$, when $t \geq t_1^+$. Then $u^{\tilde{F}} = -\psi u^F$ is a solution to (2.2) with the right-hand side $\tilde{F} = -2\partial_t u^F \partial_t \psi - u^F \partial_t^2 \psi \in C^\infty(M \times \mathbb{R})$, $\text{supp}(\tilde{F}) \subset M \times \bar{I}_1$. Thus, $\tilde{F} \in C^\infty(M \times \bar{I}_1)$ if the functions from $C^\infty(M \times \bar{I}_1)$ are identified with their zero continuations. Moreover, $(u^{\tilde{F}}(\tau_0), \partial_t u^{\tilde{F}}(\tau_0)) = (u^{-F}(\tau_0), \partial_t u^{-F}(\tau_0))$. By (BE), there are sources $F_j \in \mathcal{F}_{M,I_1}$ such that $F_j \rightarrow \tilde{F}$ in $C^\infty(M \times \bar{I}_1)$ when $j \rightarrow \infty$. Hence, (4.5) implies (4.8). The existence of a sequence $G_j \in \mathcal{F}_{M,I_1}$ which satisfies (4.9) can be seen in the same way. The case when $F \in C(\bar{U} \times \bar{I}_0)$ is proven similarly. \square

Lemma 4.5. *Using admissible measurements, for a given source $F \in \mathcal{F}_{U,I_0}$ or $F \in \mathcal{F}_{M,I_0}$, one can determine sequences of sources $F_j, G_j \in \mathcal{F}_{M,I_1}$, $j = 1, 2, \dots$, which satisfy Lemma 4.4.*

Proof. By Lemma 4.4 there exists a sequence of sources $F_j \in \mathcal{F}_{M,I_1}$, $j = 1, 2, \dots$, which satisfies (4.8). Thus, (4.1) implies that

$$\lim_{j \rightarrow \infty} E_A(F_j + F, \tau_0) = 0. \quad (4.10)$$

Now since by our assumption energy (4.10) can be evaluated using admissible measurements, we can verify if an arbitrary sequence of sources $F_j \in \mathcal{F}_{M,I_1}$ satisfies (4.10) or not. Thus, we can choose a sequence of sources $F_j \in \mathcal{F}_{M,I_1}$ which satisfies (4.10). Hence, (4.1) yields that (4.8) holds for this sequence.

By Lemma 4.6 below, the sequence $G_j \in \mathcal{F}_{M,I_1}$ which satisfies (4.9) can be determined in the same way. \square

Recall that by our assumption we can only compute energy of the sums of sources generated in the basic experiments. The following lemma shows that we can compute the energy of the difference of the sources.

Lemma 4.6. *Using admissible measurements, for any sources $F_i \in \mathcal{F}_{U_i,I_i}$, $i = 1, 2$, the energy $E_A(F_1 - F_2, \tau_0)$ can be computed at any time τ_0 , $I_i < \tau_0$, $i = 1, 2$, and also at any time $\tau_0 \in I_2$ provided $I_1 < I_2$.*

Proof. Let us prove the first part. Let $\tilde{I}_2 \in \mathcal{B}_{\mathbb{R}}$ be such that $I_i < \tilde{I}_2 < \tau_0$, $i = 1, 2$. Then by Lemma 4.5, one can determine a sequence of sources $F_j^2 \in \mathcal{F}_{M, \tilde{I}_2}$ such

$$\lim_{j \rightarrow \infty} (u^{F_j^2}(\tau_0), \partial_t u^{F_j^2}(\tau_0)) = (u^{-F_2}(\tau_0), \partial_t u^{-F_2}(\tau_0)) \quad \text{in } H^1(M, V) \times L^2(M, V).$$

Thus,

$$E_A(F_1 - F_2, \tau_0) = \lim_{j \rightarrow \infty} E_A(F_1 + F_j^2, \tau_0).$$

Since the supports of F^1 and F_j^2 are disjoint in time, the above energy can be computed using admissible measurements.

Assume now that $I_1 < I_2$ and $\tau_0 \in I_2$. Then as in the first part of the proof, fixing some arbitrary interval $\tilde{I}_1 \in \mathcal{B}_{\mathbb{R}}$ such that $I_1 < \tilde{I}_1 < I_2$, one can determine a sequence of sources $F_j^1 \in \mathcal{F}_{M, \tilde{I}_1}$ satisfying

$$\lim_{j \rightarrow \infty} E_A(F_j^1 + F_1, \tau_0) = 0.$$

Thus, one can compute

$$E_A(F_1 - F_2, \tau_0) = \lim_{j \rightarrow \infty} E_A(-F_j^1 - F_2, \tau_0) = \lim_{j \rightarrow \infty} E_A(F_j^1 + F_2, \tau_0).$$

□

Lemma 4.7. *For any sources $F_i \in \mathcal{F}_{U_i, I_i}$, $i = 1, 2$, $T > 0$ and τ_0 , $I_i < \tau_0$, $i = 1, 2$, the energy*

$$E_A(F_1 - \tau_T F_2, \tau_0)$$

can be evaluated using admissible measurements.

Proof. Let $I_3 \in \mathcal{B}_{\mathbb{R}}$ be such that $I_i < I_3 < \tau_0$, $i = 1, 2$. Then a sequence of sources $F_j^2 \in \mathcal{F}_{M, I_3}$ satisfying

$$\lim_{j \rightarrow \infty} E_A(F_j^2 + \tau_T F_2, \tau_0) = 0$$

can be determined in the same way as in Lemma 4.5. Now using admissible measurements, we can compute

$$E_A(F_1 - \tau_T F_2, \tau_0) = \lim_{j \rightarrow \infty} E_A(F_1 + F_j^2, \tau_0).$$

□

Corollary 4.8. *For any sources $F_i \in \mathcal{F}_{U_i, I_i}$, $T_i \geq 0$, $q_i \in \mathbb{Q}$, and τ_0 , $I_i < \tau_0$, $i = 1, \dots, p$, the energy $E_A(\sum_{i=1}^p q_i \tau_{T_i} F_i, \tau_0)$ can be computed using admissible measurements.*

Proof. Assume for simplicity that $q_i = m_i/n_i \geq 0$ and $l = \prod_{i=1}^p n_i$. Then $E_A(\sum_{i=1}^p q_i \tau_{T_i} F_i, \tau_0) = l^{-2} E_A(\sum_{i=1}^p \tilde{m}_i \tau_{T_i} F_i, \tau_0)$ with $\tilde{m}_i = m_i \prod_{k \neq i} n_k \in \mathbb{N}$. Now setting $\tau_{T_i^k} F_i^k = \tau_{T_i} F_i$, $k = 1, \dots, \tilde{m}_i$, we write

$$E_A(\sum_{i=1}^p \tilde{m}_i \tau_{T_i} F_i, \tau_0) = E_A(\sum_{i=1}^p \sum_{k=1}^{\tilde{m}_i} \tau_{T_i^k} F_i^k, \tau_0).$$

Taking arbitrary intervals $I_i^k \in \mathcal{B}_{\mathbb{R}}$ such that

$$I_i < I_1^1 < \dots < I_1^{\tilde{m}_1} < I_2^1 < \dots < I_2^{\tilde{m}_2} < \dots < I_p^{\tilde{m}_p} < \tau_0,$$

and using admissible measurements, one can find sequences of sources $F_j^{i,k} \in \mathcal{F}_{M, I_i^k}$ such that

$$\lim_{j \rightarrow \infty} E_A(F_j^{i,k} - \tau_{T_i^k} F_i^k, \tau_0) = 0.$$

Thus,

$$E_A(\sum_{i=1}^p q_i \tau_{T_i} F_i, \tau_0) = l^{-2} \lim_{j \rightarrow \infty} E_A(\sum_{i=1}^p \sum_{k=1}^{\tilde{m}_i} F_j^{i,k}, \tau_0).$$

□

Corollary 4.9. *For any sources $F_i \in \mathcal{F}_{U_i, I_i}$, $T_i \geq 0$, $h_i \in \mathbb{R}$, and τ_0 , $I_i < \tau_0$, $i = 1, \dots, p$, the energy $E_A(\sum_{i=1}^p h_i \tau_{T_i} F_i, \tau_0)$ can be computed using admissible measurements.*

Proof. Taking sequence $h_i^n \in \mathbb{Q}$ such that $h_i^n \rightarrow h_i$ when $n \rightarrow \infty$, we can compute

$$E_A(\sum_{i=1}^p h_i \tau_{T_i} F_i, \tau_0) = \lim_{n \rightarrow \infty} E_A(\sum_{i=1}^p h_i^n \tau_{T_i} F_i, \tau_0).$$

□

Lemma 4.10. *For any source $F \in \mathcal{F}_{U, I_0}$ and time τ_0 , $I_0 < \tau_0$, the energy $E_A(\partial_t^s F, \tau_0)$ of its s -th time derivative at time τ_0 , $s = 1, 2, \dots$, can be computed using admissible energy measurements.*

Proof. Using the forward difference approximation to the s -th order derivative and Corollary 4.8, we can compute

$$E_A(\partial_t^s F, \tau_0) = \lim_{h \rightarrow 0} \frac{1}{h^{2s}} E_A(\sum_{k=0}^s (-1)^k \binom{s}{k} \tau_{(s-k)h} F, \tau_0)$$

where $\binom{s}{k}$ are binomial coefficients which are natural numbers for all s and k .

□

Lemma 4.11. *For any sources $F, H \in \mathcal{F}_{U, I_0}$ and time τ_0 , $I_0 < \tau_0$, the energy $E_A(\partial_t^s(F - H), \tau_0)$, $s = 0, 1, 2, \dots$, can be evaluated using admissible measurements.*

Proof. The proof follows from the forward difference approximation to the s -th order derivative and Corollary 4.8. □

Using the forward difference approximation to the s -th order derivative and Corollary 4.9, we have the following lemma.

Lemma 4.12. *Let $F_i \in \mathcal{F}_{U_i, I_i}$, $i = 1, 2$. Then for any $s = 1, 2, \dots$, the energy*

$$E_A(\partial_t^s F_1 - F_2, \tau_0)$$

at any time τ_0 , $I_i < \tau_0$, can be computed using admissible measurements.

The following lemma shows that admissible measurements allow us to compute inner products between sources generated in the basic experiments and the time derivatives of waves corresponding to such sources.

Lemma 4.13. *Let $F_1 \in \mathcal{F}_{M, I_1}$ and $F_2 \in \mathcal{F}_{U, I_2}$, $-\tau < I_1 < I_2$. Then for any $\tau_0 \in I_2$, the inner product*

$$\langle F_2(\tau_0), \partial_t u^{F_1}(\tau_0) \rangle$$

can be computed using admissible measurements.

Proof. Since A is self-adjoint operator, we have

$$\begin{aligned} \partial_t E_A(F, t) &= \int_M (\langle \partial_t^2 u^F(x, t), \partial_t u^F(x, t) \rangle_x + \langle Au^F(x, t), \partial_t u^F(x, t) \rangle_x) dV(x) \\ &= \int_M \langle Au^F(x, t) + \partial_t^2 u^F(x, t), \partial_t u^F(x, t) \rangle_x dV(x) \\ &= \int_M \langle F(x, t), \partial_t u^F(x, t) \rangle_x dV(x). \end{aligned}$$

Thus, we get

$$\begin{aligned} \partial_t [E_A(F_1 + F_2, t) - E_A(F_1 - F_2, t)] &= \\ 2 \int_M (\langle F_1(t), \partial_t u^{F_2}(t) \rangle_x + \langle F_2(t), \partial_t u^{F_1}(t) \rangle_x) dV(x). \end{aligned}$$

As $I_1 < I_2$, we have

$$\text{supp } (u^{F_2}) \cap \text{supp } (F_1) = \emptyset.$$

Hence,

$$\partial_t [E_A(F_1 + F_2, t) - E_A(F_1 - F_2, t)] = 2 \langle F_2(t), \partial_t u^{F_1}(t) \rangle.$$

Thus, by our assumption and Lemma 4.6 the inner product $\langle F_2(\tau_0), \partial_t u^{F_1}(\tau_0) \rangle$ can be evaluated using admissible measurements.

□

Corollary 4.14. *Let $F_1 \in \mathcal{F}_{M,I_1}$ and $F_2 \in \mathcal{F}_{U,I_2}$, $-\tau < I_1 < I_2$. Then for any $\tau_0 \in I_2$ and $T_0, I_2 < T_0$, the inner product*

$$\langle F_2(\tau_0), \partial_t u^{F_1}(T_0) \rangle$$

can be computed using admissible measurements.

Proof. The claim follows by applying the proof of Lemma 4.13 with $\tau_{T_0-\tau_0}F_1$ instead of F_1 and the fact that $u^{\tau_{T_0-\tau_0}F_1}(\tau_0) = \tau_{T_0-\tau_0}u^{F_1}(\tau_0) = u^{F_1}(T_0)$. The latter is a consequence of (4.7).

□

The lemma below allows us to compute inner products between sources generated in the basic experiments and waves corresponding to such sources using admissible measurements.

Lemma 4.15. *Let $F_1 \in \mathcal{F}_{M,I_1}$ and $F_2 \in \mathcal{F}_{U,I_2}$, $-\tau < I_1 < I_2$. Then for any $\tau_0 \in I_2$ and $T_0, I_2 < T_0$, the inner product*

$$\langle F_2(\tau_0), u^{F_1}(T_0) \rangle$$

can be computed using admissible measurements.

Proof. Let $I_3 \in \mathcal{B}_{\mathbb{R}}$ be such that $I_1 < I_3 < I_2$. Then for $F_1 \in \mathcal{F}_{M,I_1}$, there is a sequence of sources $G_j \in \mathcal{F}_{M,I_3}$ satisfying

$$\lim_{j \rightarrow \infty} E_A(\partial_t G_j - F_1, T_0) = 0. \quad (4.11)$$

Since $I_1 < I_3 < T_0$, by Lemma 4.12 the energies in (4.11) can be computed using admissible measurements and hence, we can verify for any sequence $(G_j)_{j=1}^\infty$, $G_j \in \mathcal{F}_{M,I_3}$, if (4.11) is valid or not. Let take the sequence G_j which satisfies (4.11). Thus, (4.11) and (4.1) imply that

$$\lim_{j \rightarrow \infty} (\partial_t u^{G_j}(T_0), \partial_t^2 u^{G_j}(T_0)) = (u^{F_1}(T_0), \partial_t u^{F_1}(T_0)) \text{ in } H^1(M, E) \times L^2(M, E).$$

By Corollary 4.14 we can compute the inner products

$$\langle F_2(\tau_0), \partial_t u^{G_j}(T_0) \rangle.$$

Thus,

$$\langle F_2(\tau_0), u^{F_1}(T_0) \rangle = \lim_{j \rightarrow \infty} \langle F_2(\tau_0), \partial_t u^{G_j}(T_0) \rangle.$$

□

To reconstruct an operator A we need the following corollary which allows us to evaluate inner products between sources generated in the basic experiments and any order of time derivative of waves corresponding such sources.

Corollary 4.16. *Let $F_1 \in \mathcal{F}_{M,I_1}$ and $F_2 \in \mathcal{F}_{U,I_2}$, $-\tau < I_1 < I_2$. Then for any $\tau_0 \in I_2$, T_0 , $I_2 < T_0$ and $s \in \mathbb{N}$ the inner product*

$$\langle F_2(\tau_0), \partial_t^s u^{F_1}(T_0) \rangle$$

can be computed using admissible measurements.

4.2. Generalized sources. Let $I_0 \in \mathcal{B}_{\mathbb{R}}$. For any $s = 0, 1, 2, \dots$, let us introduce the following space of sequences of sources

$$\mathcal{F}_{I_0, \tau_0}^s = \{(F_j)_{j=1}^\infty : F_j \in \mathcal{F}_{M, I_0}, \lim_{j, k \rightarrow \infty} E_A(\partial_t^s(F_j - F_k), \tau_0) = 0\}, \quad I_0 < \tau_0.$$

By Lemma 4.11, the space $\mathcal{F}_{I_0, \tau_0}^s$ can be constructed using admissible measurements.

The definition of the space $\mathcal{F}_{I_0, \tau_0}^s$ and (4.2) imply that $(u^{F_j}(\tau_0), \partial_t u^{F_j}(\tau_0))_{j=1}^\infty$ is a Cauchy sequence in $H^{s+1}(M, V) \times H^s(M, V)$. Since $H^{s+1}(M, V) \times H^s(M, V)$ is complete, there are $a \in H^{s+1}(M, V)$ and $b \in H^s(M, V)$ such that

$$\begin{aligned} \lim_{j \rightarrow \infty} u^{F_j}(\tau_0) &= a \quad \text{in } H^{s+1}(M, V), \\ \lim_{j \rightarrow \infty} \partial_t u^{F_j}(\tau_0) &= b \quad \text{in } H^s(M, V). \end{aligned}$$

Thus, we can define an operator

$$\begin{aligned} W : \mathcal{F}_{I_0, \tau_0}^s &\rightarrow H^{s+1}(M, V) \times H^s(M, V), \\ W((F_j)_{j=1}^\infty) &= (\lim_{j \rightarrow \infty} u^{F_j}(\tau_0), \lim_{j \rightarrow \infty} \partial_t u^{F_j}(\tau_0)). \end{aligned}$$

Moreover, we can define a semi-norm on the space $\mathcal{F}_{I_0, \tau_0}^s$ of sequences of sources

$$\|(F_j)_{j=1}^\infty\|_{\mathcal{F}_{I_0, \tau_0}^s} = \lim_{j \rightarrow \infty} \sqrt{E_A(\partial_t^s F_j, \tau_0)}. \quad (4.12)$$

Lemma 4.10 allows us to compute the semi-norm (4.12).

We say that $(F_j)_{j=1}^\infty \in \mathcal{F}_{I_0, \tau_0}^s$ is equivalent to $(\tilde{F}_j)_{j=1}^\infty \in \mathcal{F}_{I_0, \tau_0}^s$ and denote by $(F_j)_{j=1}^\infty \sim (\tilde{F}_j)_{j=1}^\infty$ if

$$\begin{aligned} \lim_{j \rightarrow \infty} u^{F_j}(\tau_0) &= \lim_{j \rightarrow \infty} u^{\tilde{F}_j}(\tau_0) \quad \text{in } H^{s+1}(M, V), \\ \lim_{j \rightarrow \infty} \partial_t u^{F_j}(\tau_0) &= \lim_{j \rightarrow \infty} \partial_t u^{\tilde{F}_j}(\tau_0) \quad \text{in } H^s(M, V). \end{aligned}$$

This is equivalent to that fact that $\|(F_j - \tilde{F}_j)_{j=1}^\infty\|_{\mathcal{F}_{I_0, \tau_0}^s} = 0$. Further, we define the space $\mathcal{F}_{I_0, \tau_0}^s / \sim$ and (4.12) becomes a norm on this space. Finally, we complete $\mathcal{F}_{I_0, \tau_0}^s / \sim$ with respect to the norm (4.12), $\text{cl}(\mathcal{F}_{I_0, \tau_0}^s / \sim) = \overline{\mathcal{F}}_{I_0, \tau_0}^s$. This space is called the space of *generalized sources*.

Remark 4.17. *The space $\overline{\mathcal{F}}_{I_0, \tau_0}^s$ of generalized sources can be constructed using admissible energy measurements.*

By the definition of the completion, the space $\overline{\mathcal{F}}_{I_0, \tau_0}^s$ consists of the sequences $\widehat{F} = ((F_j^k)_{j=1}^\infty)_{k=1}^\infty$ which are the Cauchy sequences with respect to the norm (4.12). We define

$$\begin{aligned} u^{\widehat{F}}(\tau_0) &= \lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} u^{F_j^k}(\tau_0), \\ \partial_t u^{\widehat{F}}(\tau_0) &= \lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} \partial_t u^{F_j^k}(\tau_0), \\ W(\widehat{F}) &= (u^{\widehat{F}}(\tau_0), \partial_t u^{\widehat{F}}(\tau_0)). \end{aligned}$$

By the definition of the space $\overline{\mathcal{F}}_{I_0, \tau_0}^s$ the map $W : \overline{\mathcal{F}}_{I_0, \tau_0}^s \rightarrow H^{s+1}(M, V) \times H^s(M, V)$ is injective.

Lemma 4.18. $W : \overline{\mathcal{F}}_{I_0, \tau_0}^s \rightarrow H^{s+1}(M, V) \times H^s(M, V)$ is surjective.

Proof. First we show that for any $(a, b) \in C^\infty(M, V) \times C^\infty(M, V)$, there is $F \in C^\infty(M \times \overline{I}_0)$ such that $u^F(\tau_0) = a$ and $\partial_t u^F(\tau_0) = b$ where u^F is a solution to (2.2). Indeed, let v be a solution to

$$\begin{aligned} (\partial_t^2 + A)v &= 0, \quad \text{in } M \times \mathbb{R} \\ v|_{t=\tau_0} &= a, \\ \partial_t v|_{t=\tau_0} &= b. \end{aligned}$$

Let $\psi \in C^\infty(\mathbb{R})$ and $\psi(t) = 0$ if $t \leq t_0^-$ and $\psi(t) = 1$ if $t \geq t_0^+$. Then $u^F(x, t) = v(x, t)\psi(t)$ is a solution to (2.2) with $F = 2\partial_t v \partial_t \psi + v \partial_t^2 \psi \in C^\infty(M \times \mathbb{R})$, $\text{supp}(F) \subset M \times \overline{I}_0$. Thus, $F \in C^\infty(M \times \overline{I}_0)$, if the functions of $C^\infty(M \times \overline{I}_0)$ are identified with their zero continuations. Moreover, $u^F(\tau_0) = a$ and $\partial_t u^F(\tau_0) = b$.

Now let $(a, b) \in H^{s+1}(M, V) \times H^s(M, V)$. Since $C^\infty(M, V)$ is dense in all Sobolev spaces, there are $a_j, b_j \in C^\infty(M, V)$ such that

$$\begin{aligned} a &= \lim_{j \rightarrow \infty} a_j \quad \text{in } H^{s+1}(M, V), \\ b &= \lim_{j \rightarrow \infty} b_j \quad \text{in } H^s(M, V). \end{aligned}$$

By the first part of the proof, there are sources $F_j \in C^\infty(M \times \overline{I}_0)$, such that $u^{F_j}(\tau_0) = a_j$ and $\partial_t u^{F_j}(\tau_0) = b_j$. Since \mathcal{F}_{M, I_0} is dense in $C^\infty(M \times \overline{I}_0)$, for any F_j there is $F_k^j \in \mathcal{F}_{M, I_0}$ such that $\lim_{k \rightarrow \infty} F_k^j = F_j$ in $C^\infty(M \times \overline{I}_0)$. Thus, (4.5) yields that $\lim_{k \rightarrow \infty} u^{F_k^j}(\tau_0) = a_j$ in $H^{s+1}(M, V)$ and $\lim_{k \rightarrow \infty} \partial_t u^{F_k^j}(\tau_0) = b_j$ in $H^s(M, V)$. Hence, $\lim_{j, k \rightarrow \infty} u^{F_k^j}(\tau_0) = a$ in $H^{s+1}(M, V)$ and $\lim_{j, k \rightarrow \infty} \partial_t u^{F_k^j}(\tau_0) = b$ in $H^s(M, V)$. Moreover, (4.2) implies that $\lim_{k, i \rightarrow \infty} E_A(\partial_t^s(F_k^j - F_i^j), \tau_0) = 0$ and hence, $(F_k^j)_{k=1}^\infty \in \mathcal{F}_{I_0, \tau_0}^s$.

□

Now we define the space

$$\mathcal{F}_{I_0, \tau_0}^\infty = \bigcap_{s=0,1,2,\dots} \mathcal{F}_{I_0, \tau_0}^s,$$

the space of the generalized sources $\overline{\mathcal{F}}_{I_0, \tau_0}^\infty$ and the wave operator

$$W^\infty : \overline{\mathcal{F}}_{I_0, \tau_0}^\infty \rightarrow C^\infty(M, V) \times C^\infty(M, V)$$

which is bijective.

4.3. Bundle reconstruction. Let denote by δ_{x_0} the delta-distribution at $x_0 \in M$, i.e.,

$$\int_M \delta_{x_0}(x) \phi(x) dV(x) = \phi(x_0), \quad \phi \in C_0^\infty(M).$$

The main point of the bundle reconstruction is to use admissible energy measurements to find for any point $x_0 \in M$, sequences of sources that converge to delta-distributions $\lambda \delta_{x_0}$, $\lambda \in \pi^{-1}(x_0)$. Recall that we denote by m the dimension of the manifold M and then

$$\delta_{x_0} \in H^{-m/2-\epsilon}(M, V), \quad \partial^\alpha \delta_{x_0} \notin H^{-m/2-\epsilon}(M, V), \quad \epsilon \in (0, 1], \quad |\alpha| \geq 1.$$

If m is even, we set $s_0 = m/2 \in \mathbb{N}$ and otherwise, $s_0 = m/2 - 1/2 \in \mathbb{N} \cup \{0\}$.

Since M is a compact differentiable manifold, it admits a Riemannian metric. Let us denote by $B(x_0, r)$ an open ball on M at a point $x_0 \in M$ with radius r with respect to this metric. Then $B(x_0, r)$ is an open set with respect to topology on M . Let $(r_i)_{i=1}^\infty$, $r_i > 0$, be a sequence such that $r_i \rightarrow 0$ when $i \rightarrow \infty$. Then for any point $x_0 \in M$, there is a sequence of $(U_i^{x_0})_{i=1}^\infty$ of the sets $U_i^{x_0} \in \mathcal{B}_M$ such that

$$x_0 \in U_i^{x_0} \subset B(x_0, r_i).$$

For any $i \in \mathbb{N}$ and any interval $I_0 \in \mathcal{B}_\mathbb{R}$, there are sources $\mathcal{F}_{i, x_0, I_0} = \{F_{i, k} : k = 1, 2, \dots\}$ which are dense in $C(\overline{U}_i^{x_0} \times \overline{I}_0)$.

Now let $\mathcal{L}_{x_0, I_0}^{(1)}$ be a set of all sequences $(k(i), t_i)_{i \in \mathbb{N}}$ such that $k(i) \in \mathbb{N}$, $t_i \in I_0 \cap \mathbb{Q}$ and

$$\lim_{i \rightarrow \infty} F_{i, k(i)}(t_i) = \lambda \delta_{x_0} \quad \text{in} \quad H^{-s_0-1}(M, V)$$

with some section λ . Here $F_{i, k(i)} \in \mathcal{F}_{i, x_0, I_0}$.

Lemma 4.19. $\mathcal{L}_{x_0, I_0}^{(1)}$ can be constructed using admissible energy measurements.

Proof. Let an interval $I_1 \in \mathcal{B}_\mathbb{R}$ and $\tau_0 \in \mathbb{R}$ be such that $-\tau < I_1 < I_0 < \tau_0$. Then by Lemma 4.15 we can compute the inner product

$$\langle F_{i, k}(t_i), u^{\widehat{F}}(\tau_0) \rangle$$

for any $F_{i,k} \in \mathcal{F}_{i,x_0,I_0}$, $t_i \in I_0 \cap \mathbb{Q}$ and $\hat{F} \in \overline{\mathcal{F}}_{I_1,\tau_0}^{s_0}$. Let us show that $(k(i), t_i)_{i \in \mathbb{N}} \in \mathcal{L}_{x_0,I_0}^{(1)}$ if and only if there exists

$$\lim_{i \rightarrow \infty} \langle F_{i,k(i)}(t_i), u^{\hat{F}}(\tau_0) \rangle \quad (4.13)$$

for any $\hat{F} \in \overline{\mathcal{F}}_{I_1,\tau_0}^{s_0}$. Indeed, if the limit (4.13) exists, then as $W : \overline{\mathcal{F}}_{I_1,\tau_0}^{s_0} \rightarrow H^{s_0+1}(M, V) \times H^{s_0}(M, V)$ is surjective, the Banach-Steinhaus theorem implies that

$$a = \lim_{i \rightarrow \infty} F_{i,k(i)}(t_i) \quad \text{exists in } H^{-s_0-1}(M, V).$$

Since $\text{supp}(F_{i,k(i)}(t_i)) \subset U_i^{x_0} \subset B(x_0, r_i)$, $r_i \rightarrow 0$ when $i \rightarrow \infty$ and a is a distribution, we have that $\text{supp}(a) = \{x_0\}$ and thus, a is a finite linear combination of the delta-distributions at x_0 and its derivatives with coefficients from $\pi^{-1}(x_0)$, see [12, ex. 5.1.2]. By the choice of s_0 , the space $H^{-s_0-1}(M, V)$ contains delta-distributions while not their derivatives. Thus, we have $a = \lambda \delta_{x_0}$.

Hence, to construct the set $\mathcal{L}_{x_0,I_0}^{(1)}$, we determine all sequences $(k(i), t_i)_{i \in \mathbb{N}}$ such that the limits (4.13) exist. \square

Lemma 4.20. *For any $x_0 \in M$, $\mathcal{L}_{x_0,I_0}^{(1)} \neq \emptyset$.*

Proof. Let $x \rightarrow \lambda(x)$ be a section on $U_1^{x_0}$. Then consider the following sequence of functions

$$G_i(x, t) = \lambda a_i \chi_{U_i^{x_0}}(x) \chi_{I_0}(t) \in L^2(U_i^{x_0} \times I_0),$$

where

$$a_i = \frac{1}{\text{vol}(U_i^{x_0})}, \quad \text{vol}(U_i^{x_0}) = \int_{U_i^{x_0}} 1 dV(x), \quad i = 1, 2, \dots$$

As the set of sources \mathcal{F}_{i,x_0,I_0} produced in the basic experiments is dense in $L^2(U_i^{x_0} \times I_0)$, for any $\varepsilon > 0$ and $i \in \mathbb{N}$ there is $k(i) \in \mathbb{N}$, $k(i) \rightarrow \infty$ when $i \rightarrow \infty$, such that

$$\|F_{i,k(i)} - G_i\|_{L^2(U_i^{x_0} \times I_0)}^2 < \varepsilon^2 |I_0|, \quad |I_0| = \int_{I_0} 1 dt.$$

Thus, by Fubini's theorem there exists $t_i \in I_0$ such that $F_{i,k(i)}(t_i) \neq 0$ and

$$\|F_{i,k(i)}(t_i) - G_i(t_i)\|_{L^2(U_i^{x_0})} < \varepsilon. \quad (4.14)$$

As $F_{i,k(i)} \in C(\overline{I_0} \times \overline{U_i^{x_0}})$, one can find $t_i \in I_0 \cap \mathbb{Q}$ so that (4.14) holds. Hence,

$$\lim_{i \rightarrow \infty} F_{i,k(i)}(t_i) = \lim_{i \rightarrow \infty} \lambda a_i \chi_{U_i^{x_0}}(x) = \lambda \delta_{x_0} \quad \text{in } H^{-s_0-1}(M, V)$$

as $H^{s_0+1}(M, V) \hookrightarrow C(M, V)$ by Sobolev's embedding theorem. \square

Let U_{x_0} be a sufficiently small neighborhood of the point x_0 in M . Let $\mathcal{L}_{U_{x_0}, I_0}^{(2)}$ be a set of all sequences $((k(x, i), t_i)_{i \in \mathbb{N}})_{x \in U_{x_0}}$ such that

1. for any $x \in U_{x_0}$, $(k(x, i), t_i)_{i \in \mathbb{N}} \in \mathcal{L}_{x, I_0}^{(1)}$, i.e. $t_i \in I_0 \cap \mathbb{Q}$ and there are $F_{i, k(x, i)} \in \mathcal{F}_{i, x, I_0}$ such that $\lim_{i \rightarrow \infty} F_{i, k(x, i)}(t_i) = \lambda(x) \delta_x$ in $H^{-s_0-1}(M, V)$.
2. $x \rightarrow \lambda(x)$ is a C^∞ -smooth section in U_{x_0} .

Lemma 4.21. *For any $x_0 \in M$ and sufficiently small neighborhood U_{x_0} of x_0 , the set $\mathcal{L}_{U_{x_0}, I_0}^{(2)} \neq \emptyset$ and $\mathcal{L}_{U_{x_0}, I_0}^{(2)}$ can be constructed using admissible energy measurements.*

Proof. The fact that $\mathcal{L}_{U_{x_0}, I_0}^{(2)} \neq \emptyset$ for a sufficiently small neighborhood U_{x_0} of x_0 is clear.

Now it is known that λ is a C^∞ -smooth section in U_{x_0} if and only if functions

$$K_\phi : U_{x_0} \rightarrow \mathbb{R}, \quad K(x) = \langle \lambda(x), \phi(x) \rangle_x$$

are C^∞ smooth for all $\phi \in C^\infty(M, V)$. Let an interval $I_1 \in \mathcal{B}_\mathbb{R}$ and $\tau_0 \in \mathbb{R}$ be such that $I_1 < I_0 < \tau_0$. Since $W^\infty : \overline{\mathcal{F}}_{I_1, \tau_0}^\infty \rightarrow C^\infty(M, V) \times C^\infty(M, V)$ is bijective, to check whether the functions K_ϕ are C^∞ smooth for all $\phi \in C^\infty(M, V)$ is equivalent to check that the functions

$$\langle \lambda(x), u^{\hat{F}}(x, \tau_0) \rangle_x = \langle \lambda(x) \delta_x, u^{\hat{F}}(\tau_0) \rangle = \lim_{i \rightarrow \infty} \langle F_{i, k(x, i)}(t_i), u^{\hat{F}}(\tau_0) \rangle$$

are C^∞ -smooth for any generalized source $\hat{F} \in \overline{\mathcal{F}}_{I_1, \tau_0}^\infty$. The latter can be checked using admissible measurements. □

Let us denote by $\mathcal{L}_{U_{x_0}, I_0}^{(3)}$ the set consisting of all sequences $((k_1(x, i), t_i^1)_{i \in \mathbb{N}})_{x \in U_{x_0}}, \dots, ((k_d(x, i), t_i^d)_{i \in \mathbb{N}})_{x \in U_{x_0}}$ such that

1. $((k_l(x, i), t_i^l)_{i \in \mathbb{N}})_{x \in U_{x_0}} \in \mathcal{L}_{U_{x_0}, I_0}^{(2)}$, $l = 1, 2, \dots, d$, i.e. there are $F_{i, k_l(x, i)} \in \mathcal{F}_{i, x, I_0}$ such that $\lim_{i \rightarrow \infty} F_{i, k_l(x, i)}(t_i^l) = \lambda^l(x) \delta_x$ with C^∞ smooth section λ^l , $l = 1, 2, \dots, d$;
2. $\lambda^1(x_0), \dots, \lambda^d(x_0) \in \pi^{-1}(x_0)$ are linear independent;
3. d is a maximum number such that the above properties can be met.

Lemma 4.22. *For any $x_0 \in M$ and sufficiently small neighborhood U_{x_0} of x_0 , the set $\mathcal{L}_{U_{x_0}, I_0}^{(3)} \neq \emptyset$ and $\mathcal{L}_{U_{x_0}, I_0}^{(3)}$ can be constructed using energy measurements.*

Proof. The fact that $\mathcal{L}_{U_{x_0}, I_0}^{(3)} \neq \emptyset$ for a sufficiently small neighborhood U_{x_0} of x_0 is clear.

Now it is obvious that $\lambda^1(x_0), \dots, \lambda^d(x_0) \in \pi^{-1}(x_0)$ are linearly independent if and only if the functionals

$$K_{\lambda^l} : \pi^{-1}(x_0) \rightarrow \mathbb{R}, \quad K_{\lambda^l}(p) = \langle \lambda^l(x_0), p(x_0) \rangle_{x_0}, \quad l = 1, 2, \dots, d,$$

are linearly independent.

Let an interval $I_1 \in \mathcal{B}_{\mathbb{R}}$ and τ_0 be such that $-\tau < I_1 < I_0 < \tau_0$. Then by the construction of the generalized sources

$$\begin{aligned} \pi^{-1}(x_0) &= \{p(x_0) : p : M \rightarrow V \text{ is a } C^\infty \text{ smooth section}\} \\ &= \{u^{\hat{F}}(x_0, \tau_0) : \hat{F} \in \overline{\mathcal{F}}_{I_1, \tau_0}^\infty\}. \end{aligned}$$

Now the linear independence of K_{λ^l} , $l = 1, \dots, d$, is equivalent to linear independence of the functionals

$$\begin{aligned} K_l : \overline{\mathcal{F}}_{I_1, \tau_0}^\infty &\rightarrow \mathbb{R}, \quad K_l(\hat{F}) = \langle \lambda^l(x_0), u^{\hat{F}}(x_0, \tau_0) \rangle_{x_0} = \langle \lambda^l(x_0) \delta_{x_0}, u^{\hat{F}}(\tau_0) \rangle \\ &= \lim_{i \rightarrow \infty} \langle F_{i, k_l(x_0, i)}(t_i^l), u^{\hat{F}}(\tau_0) \rangle, \quad l = 1, 2, \dots, d. \end{aligned}$$

Thus, using admissible measurements one can check whether $\lambda^1(x_0), \dots, \lambda^d(x_0)$ are linearly independent. □

Lemma 4.23. *Admissible energy measurements make it possible to reconstruct local trivializations $\phi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^d$ of the vector bundle V and the $GL(d, \mathbb{R})$ -cocycle $\{t_{\alpha\beta}\}$, $t_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(d, \mathbb{R})$ between any two local trivializations.*

Proof. Let $x_0 \in M$ be an arbitrary point and choose a neighborhood $U_{x_0} \subset M$ of x_0 such that $\mathcal{L}_{U_{x_0}, I_0}^{(3)} \neq \emptyset$, i.e.

$$\lambda^l(x) \delta_x = \lim_{i \rightarrow \infty} F_{i, k_l(x, i)}(t_i^l), \quad ((k_l(x, i), t_i^l)_{i \in \mathbb{N}})_{x \in U_{x_0}} \in \mathcal{L}_{U_{x_0}, I_0}^{(3)}, \quad l = 1, 2, \dots, d.$$

As sections $\lambda^1, \dots, \lambda^d$ are C^∞ smooth, there is a neighborhood $\tilde{U} \subset U_{x_0}$ of x_0 such that $\{\lambda^1, \dots, \lambda^d\}$ is a frame field over \tilde{U} .

Let $I_1 \in \mathcal{B}_{\mathbb{R}}$ and τ_0 be such that $-\tau < I_1 < I_0 < \tau_0$. Now using admissible energy measurements, one can evaluate a local trivialization $\phi : \pi^{-1}(\tilde{U}) \rightarrow \tilde{U} \times \mathbb{R}^d$ of the vector bundle V as follows:

$$\phi(u^{\hat{F}}(x, \tau_0)) = (x, \langle \lambda^1(x), u^{\hat{F}}(x, \tau_0) \rangle_x, \dots, \langle \lambda^d(x), u^{\hat{F}}(x, \tau_0) \rangle_x), \quad \hat{F} \in \overline{\mathcal{F}}_{I_1, \tau_0}^\infty.$$

Let $U_\alpha \cap U_\beta \neq \emptyset$ and $\lambda_\alpha^1, \dots, \lambda_\alpha^d$ and $\lambda_\beta^1, \dots, \lambda_\beta^d$ be frame fields over U_α and U_β , respectively. Then for any $x \in U_\alpha \cap U_\beta$, $\lambda_\alpha^1(x), \dots, \lambda_\alpha^d(x)$ and $\lambda_\beta^1(x), \dots, \lambda_\beta^d(x)$

are bases of $\pi^{-1}(x)$. Thus, any vector $u^{\widehat{F}}(x, \tau_0) \in \pi^{-1}(x)$ can be expressed as

$$u^{\widehat{F}}(x, \tau_0) = \sum_{i=1}^d \langle \lambda_\alpha^i(x), u^{\widehat{F}}(x, \tau_0) \rangle_x \lambda_\alpha^i(x) = \sum_{i=1}^d \langle \lambda_\beta^i(x), u^{\widehat{F}}(x, \tau_0) \rangle_x \lambda_\beta^i(x).$$

Therefore, one can find a matrix $G(x) \in \text{GL}(d, \mathbb{R})$ such that

$$\begin{aligned} & (\langle \lambda_\alpha^1(x), u^{\widehat{F}}(x, \tau_0) \rangle_x, \dots, \langle \lambda_\alpha^d(x), u^{\widehat{F}}(x, \tau_0) \rangle_x) = \\ & (\langle \lambda_\beta^1(x), u^{\widehat{F}}(x, \tau_0) \rangle_x, \dots, \langle \lambda_\beta^d(x), u^{\widehat{F}}(x, \tau_0) \rangle_x) G(x) \end{aligned}$$

and hence, the $GL(d, \mathbb{R})$ -cocycle $t_{\alpha\beta}(x) = G(x)$. □

Lemma 4.23 and Theorem 2.1 imply that the bundle is reconstructed.

4.4. Reconstruction of the Riemannian structure on the vector bundle.

Let $I_0, I_1 \in \mathcal{B}_{\mathbb{R}}$ be some intervals such that $I_1 < I_0$ and $\tau_0 \in I_0$. Then consider the set

$$\mathcal{S} = \{\widehat{F} \in \overline{\mathcal{F}}_{I_1, \tau_0}^\infty : u^{\widehat{F}}(x, \tau_0) = 0 \text{ for all } x \in M\}.$$

It is clear that $\mathcal{S} \neq \emptyset$.

Lemma 4.24. *The set \mathcal{S} can be computed using admissible energy measurements.*

Proof. Let $\phi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^d$ be an arbitrary local trivialisation of the bundle V and $\lambda_\alpha^1, \dots, \lambda_\alpha^d$ be a frame field over U_α which we have already determined. Then $\widehat{F} \in \mathcal{S}$ if and only if

$$\langle u^{\widehat{F}}(x, \tau_0), \lambda_\alpha^l(x) \rangle_x = 0, \quad \text{for all } x \in U_\alpha, \quad l = 1, \dots, d, \quad \text{and all } \alpha$$

which can be verified using admissible measurements. □

Now let $\tilde{x} \in U_\alpha$ be arbitrary point and consider $B(\tilde{x}, r) \subset U_\alpha$. We use the following formula to recover the inner product of the wave $\partial_t u^{\widehat{F}}(\tilde{x}, \tau_0)$, $\widehat{F} \in \mathcal{S}$,

$$\langle \partial_t u^{\widehat{F}}(\tilde{x}, \tau_0), \partial_t u^{\widehat{F}}(\tilde{x}, \tau_0) \rangle_{\tilde{x}}^2 = \lim_{r \rightarrow 0} \frac{\|\partial_t u^{\widehat{F}}(\tau_0)\|_{L^2(B(\tilde{x}, r), V|_{B(\tilde{x}, r)})}^2}{\text{vol}(B(\tilde{x}, r))}. \quad (4.15)$$

As the manifold and measure on it are known, we can find $\text{vol}(B(\tilde{x}, r))$.

Now let us explain how to determine the L^2 -norm of the wave $\partial_t u^{\widehat{F}}(\tau_0)$ in the ball $B(\tilde{x}, r)$. First of all since $\widehat{F} \in \mathcal{S}$, by definition of energy, we can find L^2 -norm of the wave $\partial_t u^{\widehat{F}}(\tau_0)$ on the whole manifold M ,

$$\|\partial_t u^{\widehat{F}}(\tau_0)\|_{L^2(M)}^2 = 2 \lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} E_A(F_j^k, \tau_0) = 2E_A(\widehat{F}, \tau_0), \quad \widehat{F} = ((F_j^k)_{j=1}^\infty)_{k=1}^\infty.$$

Then let

$$K = K(\hat{F}, \tilde{x}, r) = \{\hat{H} \in \mathcal{S} : \partial_t u^{\hat{H}}(x, \tau_0) = \partial_t u^{\hat{F}}(x, \tau_0) \text{ for all } x \in B(\tilde{x}, r)\}.$$

The set K can be determined using energy measurements. Indeed let $\lambda_\alpha^1, \dots, \lambda_\alpha^d$ be a frame field over U_α which we have already found. Then $\hat{H} \in K$ if and only if

$$\langle \partial_t u^{\hat{F}-\hat{H}}(x, \tau_0), \lambda_\alpha^l(x) \rangle_x = 0, \quad \text{for all } x \in B(\tilde{x}, r), \quad l = 1, \dots, d.$$

By Corollary 4.14 the above conditions can be verified using admissible energy measurements.

Next

$$\|\partial_t u^{\hat{F}}(\tau_0)\|_{L^2(B(\tilde{x}, r), V|_{B(\tilde{x}, r)})}^2 = \inf_{\hat{H} \in K} \|\partial_t u^{\hat{H}}(\tau_0)\|_{L^2(M, V)}^2.$$

Thus, using (4.15) we can determine inner product of the wave $\partial_t u^{\hat{F}}(\tilde{x}, \tau_0)$, $\hat{F} \in \mathcal{S}$, with itself, for all $\tilde{x} \in U_\alpha$.

Now

$$\pi^{-1}(\tilde{x}) = \{\lambda(\tilde{x}) : \lambda \text{ is a } C^\infty \text{ smooth section}\} = \{\partial_t u^{\hat{F}}(\tilde{x}, \tau_0) : \hat{F} \in \mathcal{S}\}.$$

Hence we can recover the inner product $\langle \lambda(\tilde{x}), \lambda(\tilde{x}) \rangle_{\tilde{x}}$ for any $\lambda(\tilde{x}) \in \pi^{-1}(\tilde{x})$. Using the polarization formula for a real vector bundle, we can find

$$\begin{aligned} \langle \lambda(\tilde{x}), \tilde{\lambda}(\tilde{x}) \rangle_{\tilde{x}} &= \frac{1}{4} (\|\lambda(\tilde{x}) + \tilde{\lambda}(\tilde{x})\|_{\tilde{x}}^2 - \|\lambda(\tilde{x}) - \tilde{\lambda}(\tilde{x})\|_{\tilde{x}}^2), \\ &\text{for any } \lambda(\tilde{x}), \tilde{\lambda}(\tilde{x}) \in \pi^{-1}(\tilde{x}), \quad \tilde{x} \in U_\alpha. \end{aligned}$$

4.5. Reconstruction of operator A . To reconstruct an operator A on the vector bundle V it is enough to find its representation in an arbitrary local trivialization $\phi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^d$.

For any $\hat{F} \in \overline{\mathcal{F}}_{I_1, \tau_0}^s$, by Lemma 4.15 and Corollary 4.16 we can find the representations of $u^{\hat{F}}(x, \tau_0)$ and $\partial_t^2 u^{\hat{F}}(x, \tau_0)$ in the local trivialization. Thus, using the wave equation we can find

$$Au^{\hat{F}}(x, \tau_0) = -\partial_t^2 u^{\hat{F}}(x, \tau_0)$$

in the local trivialization. Recall that any $v \in H^{s+1}(M, V)$ can be written as $v = u^{\hat{F}}(\tau_0)$ with some $\hat{F} \in \overline{\mathcal{F}}_{I_1, \tau_0}^s$. Thus, the set of all pairs

$$\{(u^{\hat{F}}(\tau_0)|_{U_\alpha}, -u^{\partial_t^2 \hat{F}}(x, \tau_0)|_{U_\alpha}) : \hat{F} \in \overline{\mathcal{F}}_{I_1, \tau_0}^s\}$$

is the graph of the operator A in this local trivialization. This determines the operator A on M .

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K. KRUPCHYK, DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF HELSINKI,
P.O. BOX 68, FI-00014 HELSINKI, FINLAND

E-mail address: `katya.krupchyk@helsinki.fi`

M. LASSAS, DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF HELSINKI,
P.O. BOX 68, FI-00014 HELSINKI, FINLAND

E-mail address: `matti.lassas@helsinki.fi`